INTEGRATION

outcomes :
1) Area under graph
2) Volume of rotation
3) Reverse working.

$$y \underset{\text{integ}}{\overset{\text{diff}}{\rightleftarrows}} \frac{dy}{dx} \underset{\text{integ}}{\overset{\text{diff}}{\rightleftarrows}} \frac{d^2y}{dx^2}$$

HOW TO INTEGRATE :

Base Rule :

alone constant (c) $\longrightarrow$ cx

$2 \longrightarrow 2x$

$5 \longrightarrow 5x$

$8 \longrightarrow 8x$

POWER RULE: $(\square)^n \longrightarrow \dfrac{(\square)^{n+1}}{n+1}$

We are not allowed to integrate directly. We need to check conditions that need to be satisfied before we start integration.

$$\boxed{\text{Symbol: } \int \quad dx}$$

RULES:

1) YOU ARE NOT ALLOWED TO INTEGRATE POWER UNLESS "DIFFERENTIATION OF BOX" $\square'$ IS PRESENT OUTSIDE OPERATOR. (POWERS)

2) ONCE THIS CONDITION IS FULFILLED, THREE THINGS WILL DISAPPEAR,

$$\int \square' \qquad dx$$

AND POWER IS ALLOWED TO BE INTEGRATED.

$\boxed{1}$

$$\int (x+5)^7\, dx$$

$$\int 1(\boxed{x+5})^7\, dx$$

$\square = x+5$

$\square' = 1$

$$\frac{(x+5)^8}{8}$$

$\boxed{2}$

$$\int (2x+5)^7\, dx$$

$$\frac{1}{2}\int 2(\boxed{2x+5})^7\, dx$$

$\square = 2x+5$

$\square' = 2$

$$\frac{1}{\,}\ \frac{(2x+5)^8}{\,}$$

RULES:

1) YOU ARE NOT ALLOWED TO INTEGRATE POWER UNLESS "DIFFERENTIATION OF BOX" $\square'$ IS PRESENT OUTSIDE OPERATOR. (POWERS)

2) ONCE THIS CONDITION IS FULFILLED, THREE THINGS WILL DISAPPEAR,

$\int \square' \qquad dx$

AND POWER IS ALLOWED TO BE INTEGRATED.

$$2 \qquad 8$$

$$\frac{(2x+5)^8}{16}$$

3 $\int (3x+5)^4\,dx$

$$\frac{1}{3}\int 3\left(\boxed{3x+5}\right)^{\underline{\underline{4}}}dx$$

$\square = 3x+5$

$\square' = 3$

$$\frac{1}{3}\,\frac{(3x+5)^5}{5}$$

$$\frac{(3x+5)^5}{15}$$

RULES:

1) YOU ARE NOT ALLOWED TO INTEGRATE POWER UNLESS "DIFFERENTIATION OF BOX" $\square'$ IS PRESENT OUTSIDE OPERATOR. (POWERS)

2) ONCE THIS CONDITION IS FULFILLED, THREE THINGS WILL DISAPPEAR,

$$\int \square' \qquad dx$$

AND POWER IS ALLOWED TO BE INTEGRATED.

4 $\int x(4x^2+5)^6\,dx$

$$\frac{1}{8}\int 8x\left(\boxed{4x^2+5}\right)^{\underline{\underline{6}}}dx$$

$\square = 4x^2+5$

$\square' = 8x$

$$\frac{1}{8}\,\frac{(4x^2+5)^7}{7}$$

$$\frac{(4x^2+5)^7}{56}$$

$\boxed{5}$ $\int \left(\frac{1}{2}x + 3\right)^5 dx$

$2\int \frac{1}{2}\left(\boxed{\frac{1}{2}x + 3}\right)^5 dx$

$\square = \frac{1}{2}x+3$

$\square' = \frac{1}{2}$

$2\,\dfrac{\left(\frac{1}{2}x+3\right)^6}{6}$

$\dfrac{\left(\frac{1}{2}x+3\right)^6}{3} = \frac{1}{3}\left(\frac{1}{2}x+3\right)^6$

Diff	Power Peechay (multiply)	$\longrightarrow$	Minus 1 from Power	$\longrightarrow$	$\square'$ appears	
	new power is divide.	$\longleftarrow$	add 1 to Power	$\longleftarrow$	$\square'$ Disappear	$\leftarrow$ Integ

$\frac{d}{dx}\square$

$\int \square \, dx$

IMP:

P1:

INTEGRATION

WHOLE POWER ON A BRACKET

$(\Box)^n$

you have to check and complete integration conditions before integrating.

<u>RULES</u>:

1) YOU ARE NOT ALLOWED TO INTEGRATE POWER UNLESS "DIFFERENTIATION OF BOX" $\Box'$ IS PRESENT OUTSIDE OPERATOR. (POWERS)

2) ONCE THIS CONDITION IS FULFILLED, THREE THINGS WILL DISAPPEAR,

$$\int \Box' \quad dx$$

AND POWER IS ALLOWED TO BE INTEGRATED.

$$\int 7\,\boxed{x}^{3}\,dx$$

$$7\int x^3\,dx$$

$$7\frac{x^4}{4}$$

DIRECT POWER ON x

$$y = x^2 + 7x^3$$

you do not have to check for integration conditions. You can integrate directly.

$$\int 1\,\boxed{x}^{7}\,dx$$

$\Box = x$

$\Box' = 1$

$$\frac{x^8}{8}$$

$$\int (a \pm b)\,dx = \int a\,dx \pm \int b\,dx$$

$$\int (x^3 - 7x^2 + 2x)\,dx$$

Direct power on x.

Direct integrate.

$$\frac{x^4}{4} - \frac{7x^3}{3} + \frac{2x^2}{2}$$

$\int (2x+5)\,dx$ Direct Integration.

$\frac{2x^2}{2} + 5x$

(V.V. IMP) THIS CASE WILL ALWAYS HAVE POWER = 2.

Volume of Rotation.

$\int (x^2+2x)^2\,dx$

$\int (\boxed{x^2+2x})^2\,dx$

$\square = x^2+2x$

$\square' = 2x+2$

Now expand this using $(a\pm b)^2 = a^2 \pm 2ab + b^2$

we are not allowed to introduce/remove variables.

$\int (x^4 + 4x^3 + 4x^2)\,dx$

Direct power on x

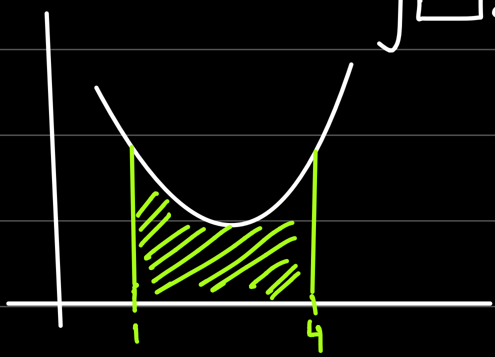
Integrate directly.

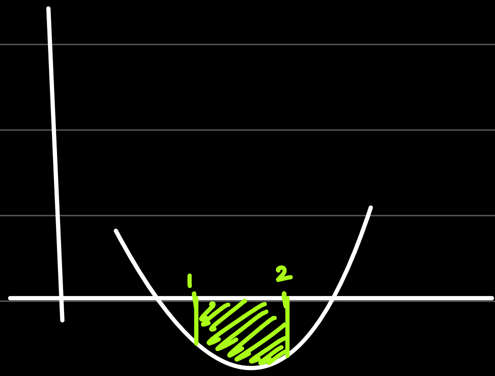
$\frac{x^5}{5} + \frac{4x^4}{4} + \frac{4x^3}{3}$

AREA UNDER GRAPH

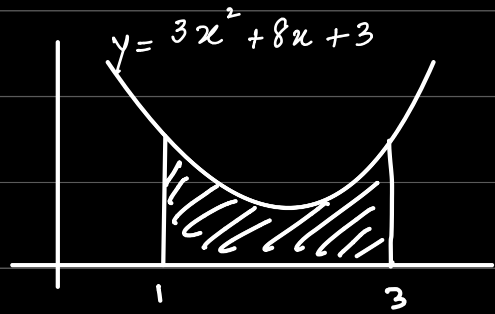
Area between curve and x-axis

$$\int \square \, dx$$





Q: $y = 3x^2 + 8x + 3$



$$\text{Area} = \int_1^3 (3x^2 + 8x + 3)\, dx$$

$$\left| \frac{3x^3}{3} + \frac{8x^2}{2} + 3x \right|_1^3$$

$$\left| \left(\frac{3(3)^3}{3} + \frac{8(3)^2}{2} + 3(3) \right) - \left(\frac{3(1)^3}{3} + \frac{8(1)^2}{2} + 3(1) \right) \right|$$

$$| 72 - 8 |$$

$$| 64 |$$

Area = 64 units^2.

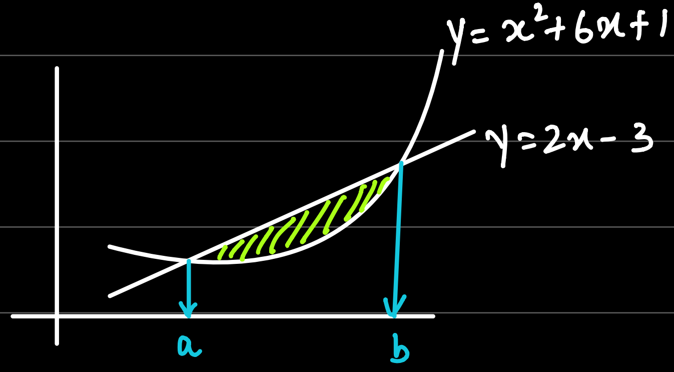
Modulus

$|x|$ = always +ve

$|3| = 3$

$|-3| = 3$

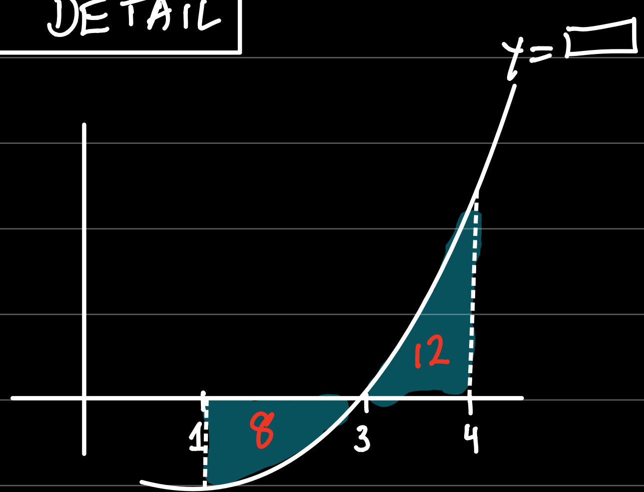
AREA BETWEEN TWO GRAPHS



Find a & b using simultaneous.

Area between two graphs $= \int_a^b (\text{upper graph})\,dx - \int_a^b (\text{lower graph})\,dx$

IMP DETAIL



Shaded region $= \int_1^3 \square\,dx + \int_3^4 \square\,dx$

$= |-8| + |12|$

$= 8 + 12$

$= 20$

$\int_1^4 \square\,dx = |-8+12| = |4| = 4$

Integration gives net area with x-axis.

$\int_1^3 \square\,dx = |-8| = 8$

$\int_3^4 \square\,dx = |12| = 12$

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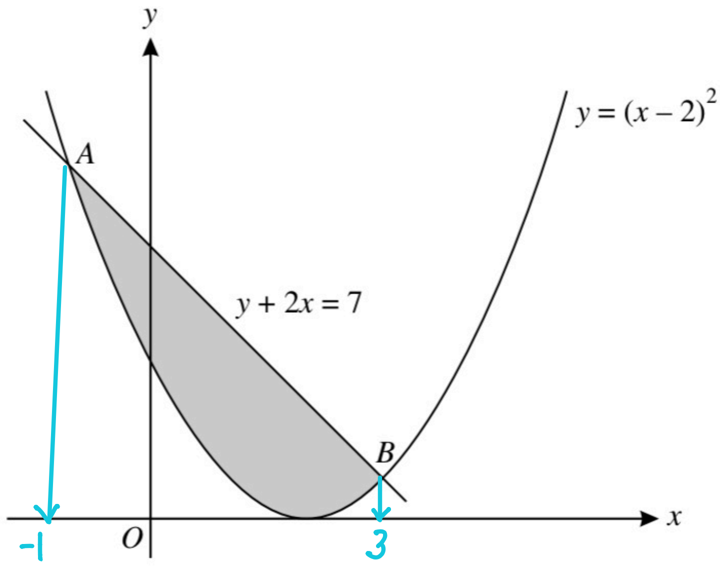


The diagram shows the curve $y = (x-2)^2$ and the line $y + 2x = 7$, which intersect at points A and B. Find the area of the shaded region. [8]

Step 1: Find A & B (simultaneously)

$y = (x-2)^2 \qquad y = 7-2x$

$(x-2)^2 = 7-2x$

$x^2 - 4x + 4 = 7 - 2x$

$x^2 - 2x - 3 = 0$

$x^2 - 3x + x - 3 = 0$

$x(x-3) + 1(x-3) = 0$

$(x+1)(x-3) = 0$

$x = -1 \quad , \quad x = 3$

upper graph:

$\int_{-1}^{3} (7-2x)\, dx$

$\left| 7x - \underline{2x^2} \right|^{3}$

lower graph.

$\int_{-1}^{3} 1\,(\boxed{x-2})^2\, dx \qquad \square = x-2 \quad \square' = 1$

$\left| \frac{(x-2)^3}{} \right|^{3}$

$\left. \quad \frac{\quad}{2} \right|_{-1}$ $\qquad$ $\left. \quad \frac{\quad}{3} \right|_{-1}$

$\left| 7x - x^2 \right|_{-1}^{3}$

$\left| \left(7(3) - (3)^2\right) - \left(7(-1) - (-1)^2\right) \right|$

$12 - (-8)$

$|20|$

$= 20$

$\left| \frac{(3-2)^3}{3} - \frac{(-1-2)^3}{3} \right|$

$\frac{1}{3} - \frac{-27}{3}$

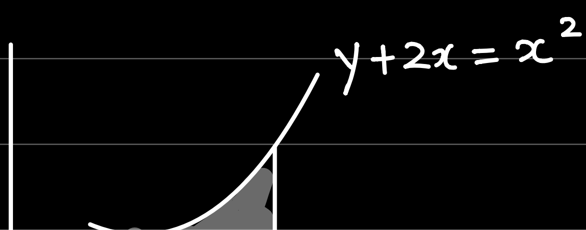
$\frac{28}{3}$

Shaded Area $= 20 - \frac{28}{3} = \frac{60-28}{3} = \boxed{\frac{32}{3}}$

VOLUME OF ROTATION (360°)

ABOUT X-AXIS	ABOUT Y-AXIS
1- MAKE Y-SUBJECT	1- MAKE X-SUBJECT
2. SQUARE BOTH SIDES	2. SQUARE BOTH SIDES
3. INTEGRATE RHS.	3. INTEGRATE RHS.
4. APPLY LIMITS	4. APPLY LIMITS
5. MULTIPLY BY π	5. MULTIPLY BY π

Q: Find the volume of rotation when the shaded region is rotated 360° about x axis.



Step 1: Make y subject $\quad y = x^2 - 2x$

2: Square both sides. $\quad y^2 = (x^2 - 2x)^2$

3. Integrate RHS $\quad \int (\boxed{x^2 - 2x})^2 \, dx.$

$\square = x^2 - 2x$

$\square' = 2x - 2$

cannot introduce variable.

expand brackets

$$\int \left[(x^2)^2 - 2(x^2)(2x) + (2x)^2 \right] dx$$

$$\int \left(x^4 - 4x^3 + 4x^2 \right) dx$$

4. Apply limits

$$\frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3}$$

$$\left| \frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right|_1^3$$

$$\left| \left(\frac{(3)^5}{5} - (3)^4 + \frac{4(3)^3}{3} \right) - \left(\frac{(1)^5}{5} - (1)^4 + \frac{4(1)^3}{3} \right) \right|$$

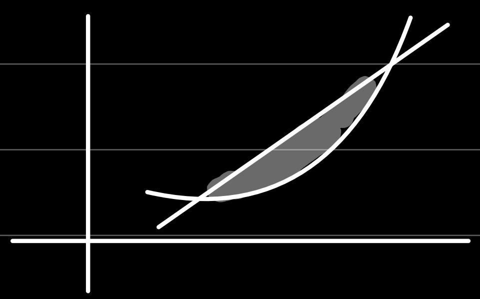
$$\left| \frac{18}{5} - \frac{8}{15} \right|$$

$$\frac{46}{15}$$

STEP 5: Multiply by π

$$\boxed{\text{VOLUME} = \frac{46}{15}\pi}$$

VOLUME BETWEEN TWO GRAPHS



Find the volume of rotation when shaded region is rotated about x-axis:

Shaded volume = Volum of upper graph − Volume of lower graph.

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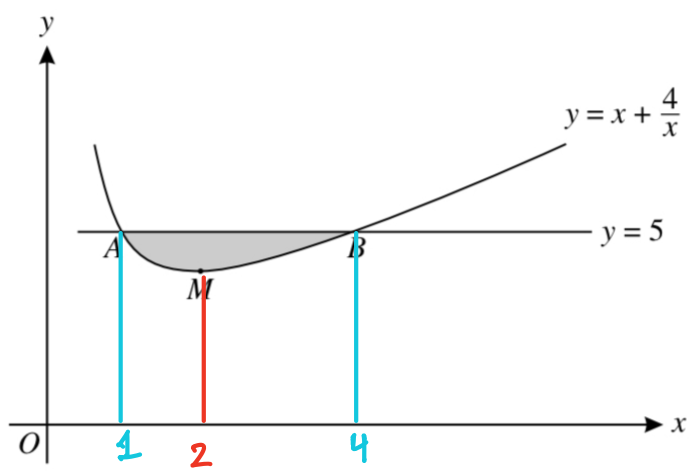

The diagram shows part of the curve $y = x + \frac{4}{x}$ which has a minimum point at M. The line $y = 5$ intersects the curve at the points A and B.

$\frac{dy}{dx} = 0$

(i) Find the coordinates of A, B and M. [5]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [6]

(i) A & B (simultaneously solve)

$y = x + \frac{4}{x}$ $\qquad$ $y = 5$

$x + \frac{4}{x} = 5$

$\frac{x^2+4}{x} = 5$

$y = x + \frac{4}{x}$

$y = x + 4x^{-1}$

$\frac{dy}{dx} = 1 + 4(-1)x^{-2}(1)$

$\frac{dy}{dx} = 1 - 4$

x

$$x^2+4=5x$$
$$x^2-5x+4=0$$
$$x^2-x-4x+4=0$$
$$x(x-1)-4(x-1)=0$$
$$(x-4)(x-1)=0$$

$x=1$, $x=4$

$y=5$, $y=5$

$A(1,5)$ $B(4,5)$

$$\frac{dy}{dx} = \quad \frac{\quad}{x^2}$$
$$0 = 1-\frac{4}{x^2}$$
$$\frac{4}{x^2} = 1$$
$$x^2=4$$
$$x=2$$
$$y=2+\frac{4}{2}=4 \quad M(2,4)$$

(About x-axis)

(iii) UPPER GRAPH

$$y=5$$
$$y^2=25$$
$$\int 25\,dx$$
$$\left|25x\right|_1^4$$
$$\left|(25(4))-(25(1))\right|$$
$$=75$$
$$\times\pi$$

Upper volume $=75\pi$

LOWER GRAPH.

$$y = x+\frac{4}{x}$$
$$y^2=\left(x+\frac{4}{x}\right)^2$$
$$\int\left(\boxed{x+\frac{4}{x}}\right)^2 dx$$

$\square = x+\frac{4}{x}$

$\square' = 1-\frac{4}{x^2}$

cannot introduce variable.

open brackets

$$\int\left((x)^2+2(x)\left(\frac{4}{x}\right)+\left(\frac{4}{x}\right)^2\right)dx$$
$$\int\left(x^2+8+\frac{16}{x^2}\right)dx$$
$$\int\left(x^2+8+16x^{-2}\right)dx$$
$$\frac{x^3}{\quad}+8x+\frac{16x^{-1}}{\quad}$$

$$3 \qquad\qquad\qquad -1$$

$$\left|\frac{x^3}{3} + 8x - \frac{16}{x}\right|_1^4$$

$$\left|\left(\frac{(4)^3}{3} + 8(4) - \frac{16}{4}\right) - \left(\frac{1^3}{3} + 8(1) - \frac{16}{1}\right)\right|$$

$$= \left|\frac{148}{3} - \left(-\frac{23}{3}\right)\right|$$

$$= 57$$

$$\times \pi$$

Lower volume $= 57\pi$

Volume of shaded region = upper Volume − Lower volume.

$$= 75\pi - 57\pi$$

$$= 18\pi$$

TYPE 3: REVERSE WORKING

$$\boxed{y \underset{\text{integ}}{\overset{\text{DiFF}}{\rightleftarrows}} \frac{dy}{dx} \underset{\text{integ}}{\overset{\text{DiFF}}{\rightleftarrows}} \frac{d^2y}{dx^2}}$$

$$y = x^2 - 4x + 2$$

DiFF $\downarrow$ INTEG $\uparrow$

$$\frac{dy}{dx} = 2x - 4$$

$$y = \int (2x-4)\,dx$$

$$y = \frac{2x^2}{2} - 4x$$

$$y = x^2 - 4x + C$$

we add +c whenever we integrate without limits

Q. $\frac{dy}{dx} = 2x - 5$

$y \xrightarrow{} \frac{dy}{dx}$

Curve $\xleftarrow{\text{integ.}} \frac{dy}{dx}$

Given that curve passes through (1,8) find equation of curve.

$y = \int (2x-5)\,dx$

$y = \frac{2x^2}{2} - 5x + c$

$\boxed{y = x^2 - 5x + c}$ $x=1$, $y=8$

$8 = 1^2 - 5(1) + c$

$c = 12$

Curve : $\boxed{y = x^2 - 5x + 12}$

22 The equation of a curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{(3x-2)}}$. Given that the curve passes through the point $P(2, 11)$, find

(i) the equation of the normal to the curve at P, (DiFF) [3]

(ii) the equation of the curve. (integ) (reverse working) [4]

(i) $\frac{dy}{dx} = \frac{6}{\sqrt{3x-2}}$

$x = 2$

(ii) $y = \int \frac{6}{\sqrt{3x-2}}\,dx$

$y = \int 6(3x-2)^{-\frac{1}{2}}\,dx$

$$m_T = \frac{6}{\sqrt{3(2)-2}} = 3$$

$$m_N = -\frac{1}{3}, \quad P(2, 11)$$

NORMAL

$$y - 11 = -\frac{1}{3}(x - 2)$$

$$3y - 33 = -x + 2$$

$$\boxed{3y = -x + 35}$$

$$y = 6\int (3x-2)^{-\frac{1}{2}}\,dx$$

$$y = \frac{6}{3}\int 3\,(\boxed{3x-2})^{-\frac{1}{2}}\,dx$$

$\square = 3x - 2$

$\square' = 3$

$$y = \frac{6}{3}\,\frac{(3x-2)^{+\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\boxed{y = 4\sqrt{3x-2} + c}$$

$x = 2$

$y = 11$

$$11 = 4\sqrt{(3)(2)-2} + c$$

$$11 = 4(2) + c$$

$$c = 3$$

$$\boxed{y = 4\sqrt{3x-2} + 3}$$

44 A curve is such that $\frac{d^2y}{dx^2} = -4x$. The curve has a maximum point at (2, 12).

$\frac{dy}{dx} = 0$, $x = 2$, $y = 12$

y curve $\xrightarrow{\text{Diff}}$ $\frac{dy}{dx}$ $\xrightarrow{\text{DIFF}}$ $\frac{d^2y}{dx^2}$; $\frac{d^2y}{dx^2}$ $\xrightarrow{\text{integ}}$ $\frac{dy}{dx}$ $\xrightarrow{\text{integ}}$ y

(i) Find the equation of the curve. [6]

$$\frac{d^2y}{dx^2} = -4x$$

integrate

$$\frac{dy}{dx} = \int -4x\,dx$$

$$\frac{dy}{dx} = \frac{-4x^2}{2} + c$$

$$\boxed{\frac{dy}{dx} = -2x^2 + c}$$

$x = 2$

$\frac{dy}{dx} = 0$

$$0 = -2(2)^2 + c$$

$$\boxed{\frac{dy}{dx} = -2x^2 + 8}$$

integrate

$$y = \int (-2x^2 + 8)\,dx$$

$$\boxed{y = -\frac{2x^3}{3} + 8x + c}$$

$x = 2$

$y = 12$

$$12 = -\frac{2(2)^3}{3} + 8(2) + c$$

$$c = 4$$

$c = 8$

$$\boxed{\frac{dy}{dx} = -2x^2 + 8}$$

3

Curve:

$$\boxed{y = -\frac{2x^3}{3} + 8x + \frac{4}{3}}$$